ISI – Bangalore Center – B Math - Physics IV – Final Exam
Date: April 29, 2010 Duration of Exam: 3 hours
Total marks:80

Answer all questions from 1 to 4 and any two from 5 to 7.

1. Please select the correct answer or answers. Each part of Q1 carries 1 point [1x5=5]

1(i). The equation for the angular part of the wave function of energy eigenstates for hydrogen atom and a 3D isotropic harmonic oscillator are
a.) same for all energy eigen states, b.) not same c.) same only for the ground state

1(ii). The quanitzed vibrational energy spectrum of HCl molecule is experimentally found to be equally spaced. The molecular vibration interaction can be modeled as
a.) harmonic oscillator . b.) a columb like central potential c.) an infinite square well potential

1(iii). For a two dimensional system with a central potential (V is a function of r only), the energy eigen spectrum is
a.) Always degenerate b.) Always non degenerate c.) Degenerate except for the ground state

1(iv). For a three dimensional central potential (e.g. the Hydrogen atom), which of the following sets of simultaneous measurements are possible? (Choose all that apply)
a.) Energy and Total angular momentum b.) Total angular momentum and one component of the angular momentum c.) Energy and one component of the angular momentum.

1(v). The one dimensional harmonic oscillator (with $V(x) = (k\, x^2)/2$) the energy eigenstates must be symmetric function of x – True or False?

2. [Total points: 20]
For one dimensional system with walls at x = 0 and x = L, the energy eigenstates and eigenvalues are given by

$$\varphi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \qquad E_n = n^2 E_1 \qquad E_1 = \frac{\hbar^2 {k_1}^2}{2m} = \frac{\hbar^2 \pi^2}{2mL^2}$$

Let the wave function at t = 0 be

$$\psi(x, 0) = \sqrt{\frac{2}{L}} \left(\frac{\sin(2\pi x/L) + 2\sin(\pi x/L)}{\sqrt{5}} \right).$$

a.) Write down the expression for $\psi(x, t)$ for all t > 0. [2 points]

b.) If $P(E_n)$ represents the probability that the measurement of energy at time t yields value E_n, then show that given the above initial state, for all t >0 [6]

$$P(E_1) = \tfrac{4}{5}$$
$$P(E_2) = \tfrac{1}{5}$$
$$P(E_n) = 0 \qquad \text{(for all other } n\text{)}.$$

c.) Show that for this initial state, expectation value of energy at time t>0 is given by

$\langle E\rangle_{t>0} = \frac{8}{5}E_1$ [4]

d.) For the same one dimensional system show that if the initial wave function $\psi(x,0)$ has the following properties:

$\psi(x,0) = 0$ for $x<0$ and $x>L$, is symmetric around $x=L/2$, and is a normalized square integrable function

then $P(E_n) = 0$ for all even n. [Hint: Use symmetry properties of energy eigenstates] [8]

3. [Total points: 15]

Consider a two dimensional Schrodinger equation $H\psi = E\,\psi$, with

$H(p_x, p_y, x,y) = H_1(p_x, x)+H_2(p_y,y)$;
$H_1(p_x, x) = ((p_x)^2/2m) + V_1(x)$, and $H_2(p_y, y) = ((p_y)^2/2m) + V_2(y)$.

a.) Show that $[H_1, H_2] = 0$ and hence find $[H, H_1]$ and $[H, H_2]$. [4]

b.) Using the above, find all the energy eigenvalues of of a two dimensional oscillator for which
$V_1(x) = (½)\, k_1\, x^2$ and $V_1(x) = (½)\, k_2\, y^2$. Are these eigenstates degenerate? [4+3]

[Use the fact that for one dimensional harmonic oscillator with potential $V(x) = (½)\, m\, \omega^2 x^2$ energy eigenvalues are given by $E_n = (n + \frac{1}{2})\hbar\omega.$]

c.) Show that if $k_1 = k_2$, Energy levels are given by $E = (h/2\pi)\,\omega\,(n+1)$. What is the degeneracy for these energy states? [4]

4. [10 points]

Show that in the *n*th eigenstate of the harmonic oscillator, the average kinetic energy $\langle T\rangle$ is equal to the average potential energy $\langle V\rangle$. That is,

$$\langle V\rangle = \frac{K}{2}\langle x^2\rangle = \langle T\rangle = \frac{1}{2m}\langle p^2\rangle$$
$$= \frac{1}{2}\langle E\rangle = \frac{\hbar\omega_0}{2}\left(n + \frac{1}{2}\right)$$

Use the following:

$$\hat{a} \equiv \frac{\beta}{\sqrt{2}}\left(\hat{x} + \frac{i\hat{p}}{m\omega_0}\right) \quad \hat{a}^\dagger = \frac{\beta}{\sqrt{2}}\left(\hat{x} - \frac{i\hat{p}}{m\omega_0}\right)$$

$$\beta^2 \equiv \frac{m\omega_0}{\hbar}$$

$$\hat{a}|n\rangle = n^{1/2}|n-1\rangle$$

$$\hat{a}^\dagger|n\rangle = (n+1)^{1/2}|n+1\rangle$$

-------------- Do ANY TWO of the following three . Each carries 15 points [2x15=30] ---------

5. Consider a one dimensional Schrodinger Equation with Potential V(x):

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x) - E]\psi;$$

a. Show that there are no bound states with Energy E less than min (V(x))
b. Prove that the bound states of the system are always non degenerate.
c. In part b) above identify where the proof breaks down in the case of non bound states and also in the case of more than one dimensions.

6. Do any three of the following: Show that

(i) the sum of two future-pointing timelike vectors is future-pointing timelike;
(ii) the sum of two future-pointing null vectors is future-pointing and either timelike or null;
(iii) every nonzero vector orthogonal to a timelike vector is spacelike;
(iv) every nonzero vector orthogonal to a null vector X is either spacelike or else a multiple of X.

Under what conditions is the sum in (ii) null?

7. Suppose that the operators $\hat{a}$ and $\hat{a}^\dagger$ in

$$\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \tfrac{1}{2})$$

obey the *anticommutation* relation

$$\{\hat{a}, \hat{a}^\dagger\} \equiv \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} = 1$$

Show that

$$\hat{a}|n\rangle = \sqrt{n}|1-n\rangle$$

$$\hat{a}^\dagger|n\rangle = \sqrt{1-n}|1-n\rangle$$

where |n> is an eigenstate of the operator $\hat{a}^\dagger\hat{a}$ with eigenvalue n.

Using positivity of H, show that there are only two energy eigenstates allowed in this system.